Superstring Loop Amplitudes from the Field Theory Limit

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(Received 23 August 2021; accepted 26 October 2021; published 16 November 2021)

We propose a procedure to determine the moduli-space integrands of loop-level superstring amplitudes for massless external states in terms of the field theory limit. We focus on the type II superstring. The procedure is to (i) take a supergravity loop integrand written in a BCJ double-copy representation, (ii) use the loop-level scattering equations to translate that integrand into the ambitwistor string moduli-space integrand, localised on the nodal Riemann sphere, and (iii) uplift that formula to one on the higher-genus surface valid for the superstring, guided by modular invariance. We show how this works for the four-point amplitude at two loops, where we reproduce the known answer, and at three loops, where we present a conjecture that is consistent with a previous proposal for the chiral measure. Useful supergravity results are currently known up to five loops.

DOI: 10.1103/PhysRevLett.127.211603

Introduction.—The birth of string theory is widely considered to be the discovery by Veneziano of the scattering amplitude formula that today bears his name [1]. More than five decades later, the calculation of string scattering amplitudes remains a formidable challenge. To give the example of the type II superstring in Minkowski spacetime, the four-point amplitude for massless external states was computed at tree level and one loop in 1982 [2,3], and at two loops in 2005 [4–6]. There has been significant work on the three-loop problem, namely, a proposal for the chiral measure [7–9] and a partial computation using the pure spinor formalism [10], but it remains to be fully addressed. The advances have had a rich interplay with those in gauge theory and gravity amplitudes, particularly in their maximally supersymmetric versions. For instance, the first computations of the four-point one-loop amplitudes in the now widely studied 4D $\mathcal{N}=4$ super-Yang-Mills theory (SYM) and $\mathcal{N}=8$ supergravity were based on the field theory limit of the analogous superstring calculations [11]. In this Letter, we aim to return the favor by importing three-loop results in $\mathcal{N}=8$ supergravity, themselves obtained from nonplanar $\mathcal{N}=4$ SYM via the Bern-Carrasco-Johansson (BCJ) double copy [12], into the type II superstring.

String theory versus field theory.—We will consider the type II superstring four-point amplitude for massless incoming states of momenta k_i ($i=1,\dots,4$). The 10D maximal supersymmetry implies that information on the four external states is encoded in a kinematic prefactor $\mathcal{R}^4$ [13], such that the supergravity tree-level amplitude is $\sim\mathcal{R}^4/(s_{12}s_{13}s_{14})$. We define the Mandelstam variables as $s_{ij}=2k_i\cdot k_j$. Our working assumption will be that, up to three loops [14], the g-loop superstring amplitude $\mathcal{A}_{\mathbb{S}}^{(g)}$ takes the form

$$\frac{\mathcal{A}_{\mathbb{S}}^{(g)}}{\mathcal{R}^4}=\int_{\mathcal{M}_{g,4}}\left|\prod_{I\leq J}d\Omega_{IJ}\right|^2\int d\ell|\mathcal{Y}_{\mathbb{S}}^{(g)}|^2\prod_{i<j}|E(z_i,z_j)|^{\frac{\alpha' s_{ij}}{2}}\times\left|\exp\frac{\alpha'}{2}\left(i\pi\Omega_{IJ}\ell^I\cdot\ell^J+2\pi i\sum_j\ell^I\cdot k_j\int_{z_0}^{z_j}\omega_I\right)\right|^2. \tag{1}$$

The integration denoted by $\mathcal{M}_{g,4}$ is over a genus-g fundamental domain parametrized by the period matrix Ω_{IJ} ($I,J=1,\dots,g$) and over four marked points z_i. We use a "chiral splitting" representation [16,17], made possible by the introduction of the loop momenta ℓ^I, with $d\ell$ denoting $\prod_I d^{10}\ell^I$. The appearance of the prime form $E(z_i,z_j)$ and the exponential (involving the holomorphic Abelian differentials ω_I whose cycles define the period matrix) constitute the chiral $\times$ antichiral loop-level Koba-Nielsen factors. The interesting object is $\mathcal{Y}_{\mathbb{S}}^{(g)}$. We make no distinction between type IIA and type IIB apart from the details of $\mathcal{R}^4$, since at four points there is no contribution from odd spin structures at least up to three loops [18].

We will exploit the analogy between the formula (1) for the superstring and the following expected formula for supergravity:

$$\frac{\mathcal{A}_{\mathbb{A}}^{(g)}}{\mathcal{R}^4} = \int d\ell \int_{\mathcal{M}_{g,4}} \prod_{I\leq J} d\Omega_{IJ} (\mathcal{Y}_{\mathbb{A}}^{(g)})^2 \prod_{i=1}^{4} \bar{\delta}(\mathcal{E}_i) \prod_{I\leq J} \bar{\delta}(u^{IJ}). \quad (2)$$

This type of formula for a scattering amplitude was discovered at tree level by Cachazo, He, and Yuan [20,21], generalizing a previous formula from twistor string theory [22,23]. The loop-level extension [24–30] was derived from the type II ambitwistor string [31], which is a worldsheet model of type II supergravity. The 10D loop integration in Eq. (2) is UV divergent, so the expression is formal only, and we understand it as defining a loop integrand. The genus-g moduli-space integration is fully localized on a set of critical points, determined by the genus-g *scattering equations*: $\mathcal{E}_i = 0$ and $u^{IJ} = 0$ [32]. An extensive discussion of the loop-level version of this formalism was presented in Ref. [30]; the brief discussion below will be sufficient for our purposes. There is a clear analogy between Eqs. (1) and (2). Our proposal, under conditions to be discussed, is to identify the "chiral half-integrands,"

$$\mathcal{Y}_{\mathbb{S}}^{(g)} = \mathcal{Y}_{\mathbb{A}}^{(g)}, \quad (3)$$

which is known to be possible for $g \leq 2$. Notice that Eq. (1) is a simplified expression where $\mathcal{Y}_{\mathbb{S}}^{(g)}$ is independent of α'. The idea is that we can import an ambitwistor string—i.e., supergravity—result into the superstring.

The only known procedure to evaluate Eq. (2) reflects the fact that the ambitwistor string is a field theory in disguise: the genus-g formula can be localized on a maximal nonseparating degeneration, i.e., a Riemann sphere with g nodes, as in Fig. 1. This follows from a residue argument in moduli space at one [27,28] and two [29,30] loops, and our three-loop results provide evidence that it holds at higher order. The formula on the nodal sphere is

$$\frac{\mathcal{A}_{\mathbb{A}}^{(g)}}{\mathcal{R}^4} = \int \frac{d\ell}{\prod_I (\ell^I)^2} \int_{\mathcal{M}_{0,4+2g}} c^{(g)} (\mathcal{J}^{(g)} \mathcal{Y}^{(g)})^2 \prod_{A=1}^{4+2g} \bar{\delta}(\mathcal{E}_A). \quad (4)$$

Here, $\mathcal{M}_{0,4+2g}$ is the moduli space of the Riemann sphere with $4+2g$ marked points, corresponding to 4 external particles and $2g$ "loop marked points," one pair per node as in Fig. 1. The factors $c^{(g)}$ and $\mathcal{J}^{(g)}$ arise from the

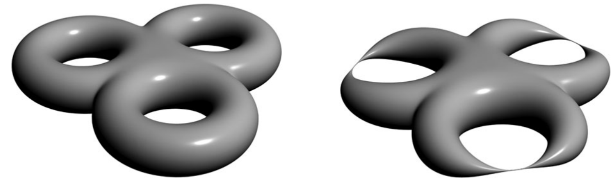

FIG. 1. Genus-3 surface and its maximal nonseparating degeneration (genus 0) with 2 marked points per node.

degeneration of $\mathcal{M}_{g,4}$ to $\mathcal{M}_{0,4+2g}$ [30]. We will give an example momentarily. The object $\mathcal{Y}^{(g)}$ in this expression is the limit of $\mathcal{Y}_{\mathbb{A}}^{(g)}$ in the maximal nonseparating degeneration. Finally, the delta functions impose the loop-level scattering equations on the nodal sphere, $\mathcal{E}_A = 0$, on whose finite set of solutions the moduli-space integral fully localizes; in fact, this integral can be understood as a multidimensional residue integral.

Let us be more concrete. The degeneration to the g-nodal sphere is achieved in a limit involving the diagonal components of the period matrix: $q_{II} = e^{i\pi\Omega_{II}} \to 0$. In this limit, the holomorphic Abelian differentials whose periods define the period matrix acquire simple poles at the corresponding node: with $\sigma \in \mathbb{CP}^1$,

$$\omega_I = \frac{\omega_{I^+I^-}}{2\pi i}, \qquad \omega_{I^+I^-}(\sigma) = \frac{(\sigma_{I^+} - \sigma_{I^-})d\sigma}{(\sigma - \sigma_{I^+})(\sigma - \sigma_{I^-})}, \quad (5)$$

where the $\sigma_{I^\pm}$ are the marked points for node I. Together with the marked points σ_i associated to the four external particles, we have the total of $4+2g$ marked points parametrizing $\mathcal{M}_{0,4+2g}$ up to $\mathrm{SL}(2,\mathbb{C})$. For $g \geq 2$, the off-diagonal components of the period matrix are expressed in this limit in terms of cross ratios of the nodal marked points,

$$q_{IJ} = e^{2i\pi\Omega_{IJ}} = \frac{\sigma_{I^+J^+}\sigma_{I^-J^-}}{\sigma_{I^+J^-}\sigma_{I^-J^+}}, \quad (6)$$

where we denote $\sigma_{AB} = \sigma_A - \sigma_B$. This change of integration variables leads to the $(\mathcal{J}^{(g)})^2$ appearing in Eq. (4). One $\mathcal{J}^{(g)}$ arises from the moduli-space measure,

$$\prod_{I<J} \frac{dq_{IJ}}{q_{IJ}} = \frac{\mathcal{J}^{(g)}}{\mathrm{vol}\ \mathrm{SL}(2,\mathbb{C})}, \qquad \mathcal{J}^{(g)} = J^{(g)} \prod_{I^\pm} d\sigma_{I^\pm}, \quad (7)$$

while the other arises from rewriting higher-genus scattering equations as nodal sphere ones. Finally, the scattering equations on the nodal sphere are equivalent to the vanishing of a meromorphic quadratic differential $\mathfrak{P}^{(g)}$ with only simple poles, and can be read off from the residues of this differential at the $4+2g$ marked points,

$$\mathcal{E}_A = \mathrm{Res}_{\sigma_A} \mathfrak{P}^{(g)}. \tag{8}$$

The ingredients of Eq. (4) can be illustrated with the two-loop example. We have $c^{(2)} = 1/(1 - q_{12})$ [33] and

$$\mathfrak{P}^{(2)} = P^2 - (\ell^I \omega_{I^+ I^-})^2 + (\ell_1^2 + \ell_2^2)\omega_{1^+ 1^-} \omega_{2^+ 2^-}, \tag{9}$$

where

$$P_\mu(\sigma) = \ell_\mu^I \omega_{I^+ I^-}(\sigma) + \sum_i \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma. \tag{10}$$

Effectively, $\mathfrak{P}^{(g)}$ encodes all the potential loop-integrand propagators in an expression like Eq. (4), while $c^{(g)}$ projects out certain unphysical propagators. These details are not important for this Letter, where we are concerned with $\mathcal{J}^{(g)}$ and especially $\mathcal{Y}^{(g)}$. At two loops, we have

$$J^{(2)} = \frac{1}{\sigma_{1^+ 2^+} \sigma_{1^+ 2^-} \sigma_{1^- 2^+} \sigma_{1^- 2^-}} \tag{11}$$

and

$$\mathcal{Y}^{(2)} = \frac{1}{3}[(s_{14} - s_{13})\Delta_{12}^{(2)} \Delta_{34}^{(2)} + \mathrm{cyc}(234)], \tag{12}$$

where we used the determinant

$$\Delta_{i_1 \ldots i_g}^{(g)} = \varepsilon^{I_1 \ldots I_g} \omega_{I_1}(\sigma_{i_1}) \ldots \omega_{I_g}(\sigma_{i_g}) \tag{13}$$

defined for any g. The expression (12) is built from the differentials ω_I, which naturally lift from the nodal sphere to become the holomorphic Abelian differentials on the genus-2 surface. Indeed, the genus-2 expression is also valid as $\mathcal{Y}_{\mathbb{A}}^{(2)}$ in Eq. (2) and, crucially for us, as $\mathcal{Y}_{\mathbb{S}}^{(2)}$ in Eq. (1). The object $\Delta^{(g)}$ is a modular form of weight -1 at any genus, which at genus 2 gives $\mathcal{Y}_{\mathbb{S}}^{(2)}$ the appropriate weight such that the moduli-space integral is well defined. At three loops, the answer is not as simple as Eq. (12): $\Delta^{(3)}$ still arises [10], but additional ingredients are needed, as discussed, e.g., in Ref. [34], and as we will see here.

$\mathcal{Y}_{\mathbb{S}}^{(g)}$ from BCJ numerators.—Let us present and test our strategy. The steps are to (i) take a supergravity loop integrand written in a BCJ double-copy representation, (ii) translate that integrand into the ambitwistor string moduli-space integrand localized on the nodal Riemann sphere, i.e., obtain $\mathcal{Y}^{(g)}$, (iii) uplift that formula to a higher-genus modular form conjecturally valid for the superstring, i.e., obtain $\mathcal{Y}_{\mathbb{S}}^{(g)}$ such that $\mathcal{Y}_{\mathbb{S}}^{(g)} \to \mathcal{Y}^{(g)}$ as $q_{II} \to 0$.With our current understanding, step (iii) relies on an educated guess, as we will exemplify.

Starting with step (i), a BCJ representation is one in which the loop integrand is written in terms of trivalent diagrams, whose numerators are the square of analogous numerators in nonplanar SYM obeying the BCJ color-kinematics duality [12,35] [36]. See Ref. [47] for a review of this remarkable construction, which was motivated by the KLT relations of string theory [48]. Indeed, there is a large body of work relating this construction to aspects of string theory, e.g., Refs. [49–65]. Step (ii) is based on the connection to the scattering equations story, for which we use the following relation based on a differential form with logarithmic singularities [66]

$$(2\pi i)^4 \mathcal{J}^{(g)} \mathcal{Y}^{(g)} = \sum_{\rho \in S_{2+2g}} \frac{N^{(g)}(1^+, \rho, 1^-)}{(1^+, \rho, 1^-)} \prod_{A=1}^{4+2g} d\sigma_A, \tag{14}$$

where $(ABC\ldots D) = \sigma_{AB}\sigma_{BC}\ldots\sigma_{DA}$ is a Parke-Taylor denominator. The BCJ numerators $N^{(g)}$, which depend on a particle ordering, are SYM numerators whose square gives the supergravity numerators; this square effectively translates into the square of $\mathcal{J}^{(g)}\mathcal{Y}^{(g)}$ in Eq. (4). Notice, however, that we have extracted the overall factor $\mathcal{R}^4$ in Eq. (4), whose "square root" is therefore not included in the SYM numerators. The correspondence between the numerators $N^{(g)}$ and trivalent diagrams is best understood in an explicit example, to be discussed below. Before that, let us make two comments. The first is that two marked points singled out in Eq. (14) were chosen to be $\sigma_{1^\pm}$, but the sum is independent of that choice. The second, for the reader familiar with the scattering equations formalism including the developments [70–73], is that equalities like (14) often hold only when the marked points satisfy the scattering equations (e.g., for CHY Pfaffians). Here, on the other hand, we propose that Eq. (14) defines $\mathcal{Y}^{(g)}$ such that it may be uplifted to the superstring, as happens up to two loops.

Let us test the strategy at two loops, for which the BCJ representation of the four-point supergravity loop integrand is long known [74,75]. The two-loop BCJ numerators can be compactly written as

$$N^{(2)}(1^+, \rho_1, 2^\pm, \rho_2, 2^\mp, \rho_3, 1^-) = \begin{cases} s_{ij} & \rho_2 = \{i, j\} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

They correspond to half-ladder diagrams with loop momenta $\pm\ell_1$ at the ends; see Fig. 2. A standard two-loop diagram is then obtained by gluing the nodal legs, i.e., I^+ with I^-. Taking the result (15) from the literature, it is possible to obtain $\mathcal{Y}^{(2)}$ via Eq. (14). Then, it is both natural

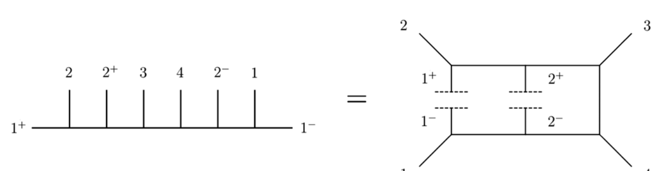

FIG. 2. Two-loop example. Diagram associated with the numerator $N(1^+, 2, 2^+, 3, 4, 2^-, 1, 1^-)$.

and easy to rewrite $\mathcal{Y}^{(2)}$ in the form (12), which, as explained earlier, can be uplifted to genus 2, matching the superstring result $\mathcal{Y}^{(2)}_{\mathbb{S}}$. This achieves step (iii).

Three loops.—We now apply our strategy to the much more intricate three-loop case. From the general form of a three-loop field theory integrand, namely, the inclusion of the relevant diagram topologies, we can determine $c^{(3)}$ and $\mathfrak{P}^{(3)}$. However, they do not appear in Eq. (14), so they are not important for the goal of this Letter [76]. The important quantities are $\mathcal{J}^{(3)}$ and $\mathcal{Y}^{(3)}$. The Jacobian is straightforwardly obtained from Eq. (7) and can be written as

$$J^{(3)} = J_{\text{hyp}} \frac{\prod_I \sigma_{I^+ I^-}}{\prod_{I<J} \sigma_{I^+ J^+} \sigma_{I^- J^-} \sigma_{I^+ J^-} \sigma_{I^- J^+}}, \tag{16}$$

where in the factor

$$J_{\text{hyp}} = \sigma_{1^+ 2^-} \sigma_{2^+ 3^-} \sigma_{3^+ 1^-} - \sigma_{1^+ 3^-} \sigma_{3^+ 2^-} \sigma_{2^+ 1^-} \tag{17}$$

the subscript refers to *hyperelliptic*, as we will explain.

We can now determine $\mathcal{Y}^{(3)}$ using Eq. (14). The right-hand side is obtained from the known BCJ representation of the three-loop supergravity integrand, a landmark application of the double copy [12,77]. The BCJ numerators, listed in Table I of Ref. [12], are not as simple as at two loops and depend linearly on the loop momenta, e.g., [78]

$$N(1^+,1,2,2^+,3,3^+,2^-,4,3^-,1^-) = \frac{1}{3} s_{12}(s_{12} - s_{14}) + \frac{2}{3} \ell^1 \cdot [k_2(s_{13} - s_{14}) + k_3(s_{13} - s_{12}) + k_4(s_{12} - s_{14})].$$

Via Eq. (14), this property implies

$$2\pi i \mathcal{Y}^{(3)}_{\mathbb{S}} = \mathcal{Y}_0 + 2\pi i \ell^I_\mu \mathcal{Y}^\mu_I, \tag{18}$$

where the factors were chosen for later convenience. We write our results already in uplifted form, i.e., for $\mathcal{Y}^{(3)}_{\mathbb{S}}$ (which we claim is $\mathcal{Y}^{(3)}_{\mathbb{A}}$) instead of its degeneration $\mathcal{Y}^{(3)}$. To determine $\mathcal{Y}^{(3)}_{\mathbb{S}}$, we construct a well-motivated ansatz with the required modular weight of -1, and fix the coefficients of that ansatz by matching numerically the degeneration limit to Eq. (14). This requires expanding in the degeneration parameters the Jacobi theta functions which define various objects, a straightforward if computationally heavy procedure.

The second term in Eq. (18) is the easiest: we can write

$$\mathcal{Y}^\mu_I = \frac{2}{3} [\alpha^\mu_1 \omega_I(z_1) \Delta^{(3)}_{234} + \text{cyc}(1234)], \tag{19}$$

with $\alpha^\mu_1 = k^\mu_2 (k_3 - k_4) \cdot k_1 + \text{cyc}(234)$. All the ingredients have been introduced previously.

The object $\mathcal{Y}_0$ is more involved. It is convenient to extricate the kinematic dependence by writing

$$\mathcal{Y}_0 = s_{13} s_{14} Y_{12,34} + \text{cyc}(234), \tag{20}$$

where $Y_{12,34}$ is independent of the s_{ij} and is symmetric when exchanging: $z_1 \leftrightarrow z_2$, $z_3 \leftrightarrow z_4$, $\{z_1, z_2\} \leftrightarrow \{z_3, z_4\}$. Let us first state the result and then discuss it:

$$Y_{12,34} = \frac{1}{3} \mathcal{D}_{12,34} - \frac{1}{15\Psi_9} \left(\mathcal{S}^{(a)}_{12,34} - \frac{1}{8} \mathcal{S}^{(b)}_{12,34} \right), \tag{21}$$

where

$$\mathcal{D}_{12,34} = \omega_{3,4}(z_1) \Delta^{(3)}_{234} + \omega_{3,4}(z_2) \Delta^{(3)}_{134} + \omega_{1,2}(z_3) \Delta^{(3)}_{412} + \omega_{1,2}(z_4) \Delta^{(3)}_{312}, \tag{22}$$

$$\mathcal{S}^{(a)}_{12,34} = \sum_\delta \Xi_8[\delta] [S_\delta(z_1,z_2) S_\delta(z_2,z_3) S_\delta(z_3,z_4) S_\delta(z_4,z_1) + S_\delta(z_2,z_1) S_\delta(z_1,z_3) S_\delta(z_3,z_4) S_\delta(z_4,z_2)], \tag{23}$$

$$\mathcal{S}^{(b)}_{12,34} = \sum_\delta \Xi_8[\delta] S_\delta(z_1,z_2)^2 S_\delta(z_3,z_4)^2. \tag{24}$$

Starting with the expression (22), the object $\omega_{i,j}(z_k)$ is the normalized Abelian differential of the third kind, whose degeneration limit is

$$\omega_{i,j}(\sigma) = \frac{\sigma_{ij}}{(\sigma - \sigma_i)(\sigma - \sigma_j)} d\sigma. \tag{25}$$

A consistency check is that the contribution (22), including the kinematic coefficient, is completely fixed by Eq. (19). This follows from the condition of "homology invariance": distinct choices of homology cycles of the Riemann surface with respect to the marked points z_i obey monodromy relations dictated by the chiral splitting procedure [17], and this connects the two contributions [79].

The contributions (23) and (24) are more elaborate, but the structure is familiar from the RNS formalism [4,16,81–86]. The sums are over the 36 even spin structures at genus 3, labeled by δ, and the objects $S_\delta(z_i, z_j)$ are the Szegő kernels arising from the OPEs of worldsheet fermions. The "chiral measure" $\Xi_8[\delta]/\Psi_9$ is the crucial ingredient. Here, $\Psi_9 = \sqrt{-\prod_\delta \theta[\delta](0)}$ is a modular form of weight 9 (note our nonstandard definition for the sign), defined in terms of the even Jacobi theta functions.

The general properties of the chiral measure were described in Refs. [7,8] and the precise definition of $\Xi_8[\delta]$ was given in Ref. [9]. It is a sophisticated definition, so we will not repeat it here; we found Ref. [87] very helpful. The RNS derivation of this measure remains obscure; see Appendix C of Ref. [88].

In the degeneration limit $q_{II} \to 0$, Ψ_9 vanishes with leading behavior $\Psi_9 = (\prod_I q_{II}^2)\psi_9 + \cdots$,

$$\psi_9 = 2^{14} J_{\text{hyp}} \frac{(\prod_I \sigma_{I^+I^-})^3}{\prod_{I<J} \sigma_{I^+J^+}\sigma_{I^-J^-}\sigma_{I^+J^-}\sigma_{I^-J^+}}, \qquad (26)$$

where J_{hyp} is given in Eq. (17). It is opportune to note that only a codimension-$1_{\mathbb{C}}$ subset of genus-3 Riemann surfaces are hyperelliptic (whereas for $g \leq 2$ all surfaces are), and these are precisely identified by the vanishing of Ψ_9 [89]. The condition $J_{\text{hyp}} = 0$ identifies hyperelliptic surfaces in the degeneration limit. The factors of J_{hyp} in $J^{(3)}$ and in $1/\Psi_9$ cancel, such that $\mathcal{J}^{(3)}\mathcal{Y}^{(3)}$ does not vanish in the hyperelliptic sector.

The sums (23) and (24), which are modular forms of weight 8, vanish in the degeneration limit in a manner analogous to Ψ_9, so that the ratio appearing in Eq. (21) yields a finite result on the nodal sphere [91]. As consistency checks on our implementation of the chiral measure, we verified to order $O(q_{II}^2)$ the following identities (respectively, from Refs. [9,92,93]):

$$\sum_\delta \Xi_8[\delta] = 0, \qquad \sum_\delta \Xi_8[\delta] S_\delta(z_1, z_2)^2 = 0,$$
$$\sum_\delta \Xi_8[\delta] S_\delta(z_1, z_2) S_\delta(z_2, z_3) S_\delta(z_3, z_1) = C\Psi_9 \Delta_{123}^{(3)},$$

where we determined the previously unknown coefficient $C = 15(2\pi i)^3$. We could not find simplified expressions for Eqs. (23) and (24); they are not proportional to Ψ_9, i.e., not proportional to J_{hyp} in the degeneration limit.

Comparing our result to the pure spinor computation of Ref. [10], the latter was restricted to part of the correlator and was not manifestly modular invariant, but appears to be consistent at least with Eq. (19). The main goal of Ref. [10], for which the partial computation was sufficient, was to match a prediction from S duality [94] for the low-energy amplitude, where the overall normalization is important. We neglected the normalization here, and leave this aspect and a proper comparison to Ref. [10] for future work. Because of manifest supersymmetry, the splitting of spin structures does not arise in the pure spinor approach [5,6,95–98], so this approach may be helpful in simplifying the sums seen above.

Discussion.—We have constructed a conjectured expression for the three-loop four-point amplitude of massless states in the type II superstring. The crucial ingredient is the chiral half-integrand (18). As at two loops [4,80], this object can also, in principle, be imported into the Heterotic superstring, paired with a bosonic counterpart.

In place of a first-principles worldsheet calculation, we wrote down an ansatz inspired by insights from the RNS and pure spinor formalisms, and then constrained that ansatz using supergravity data mined with modern amplitudes techniques. Our focus was on briefly delineating a strategy, with very concrete results. Additional technical details will be presented elsewhere. We hope that our conjecture can guide rigorous derivations using established worldsheet methods. Alternatively, in the spirit of the amplitudes program, perhaps the proof can follow from a set of basic constraints, such as unitarity.

Natural future directions are the study of the moduli-space integration in the low-energy limit, building on Refs. [10,99–102], which is newly motivated by beautiful advances in the nonperturbative amplitudes bootstrap [103]; and the consideration of higher-point [7,9,112–118] amplitudes. We expect our strategy to prove useful, not least because there are BCJ numerators for $\mathcal{N} = 8$ supergravity up to five loops [119–121], although the five-loop case required a generalization of this representation. Also at this loop order, the relation between supermoduli space and ordinary moduli space becomes more intricate [15], calling into question the structure of our starting point (1). The interplay between field theory and string theory amplitudes continues to present us with many challenges and fruitful surprises.

We thank Eric D'Hoker, Carlos Mafra, Boris Pioline, Rodolfo Russo, Oliver Schlotterer, and Edward Witten for comments. Y. G. is supported by the CUniverse research promotion project "Toward World-class Fundamental Physics" of Chulalongkorn University (Grant No. CUAASC). R. M. and R. S. M. are supported by the Royal Society via a University Research Fellowship and a Studentship Grant, respectively.

Note added.—Recently, it came to our attention that the authors of Ref. [80] have independently constructed the contribution to the half-integrand that is linear in the loop momenta, Eq. (19).

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[76] They will be discussed elsewhere.
[77] The result in Ref. [12] applies to 4D $\mathcal{N} = 8$ supergravity, but we will assume that it "oxidates" trivially to 10D type II supergravity for similar reasons as in the two loop case, given the absence of contributions from odd spin structures.

[78] Our convention for the external momenta is that they are incoming, whereas the convention in Ref. [12] was that they are outgoing. This affects the sign of the term linear in the loop momenta.

[79] In summary, if a marked point z_i is shifted by a "B cycle," (i) the loop momentum associated to that cycle is shifted by k_i and (ii) the Abelian differential of the third kind has nontrivial monodromy. These two effects combine precisely to achieve homology invariance. See the very clear discussion for the two-loop five-point amplitude in Ref. [80], where the objects $g^I_{i,j}$ relate to $\omega_{3,4}(z_1)$ as $\omega_{3,4}(z_1) = (g^I_{1,3} - g^I_{1,4})\omega_I(z_1)$.

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